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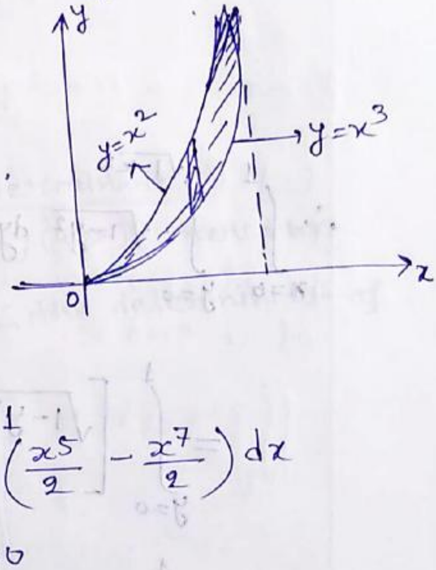
B.Sc.- III

MATHEMATICS HONS : Paper- V

group- B. (Multiple Integrals)

Contents:→ Multiple integral

Example① Evaluate $\int_{x=0}^{1}\int_{y=x^3}^{y=x^2} xy\,dy\,dx$. Sketch the region D for which this iterated integral gives the double integral $\iint_D xy\,dx\,dy$.



Solution :→

Here y ranges x^3 to x^2, while x goes from 0 to 1.

$$\therefore \int_{x=0}^{1}\int_{y=x^3}^{x^2} xy\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\frac{xy^2}{2}\right]_{y=x^3}^{x^2} dx = \int_{x=0}^{1}\left(\frac{x^5}{2}-\frac{x^7}{2}\right)dx$$

$$= \left[\frac{x^6}{12}-\frac{x^8}{16}\right]_{x=0}^{1} = \frac{1}{12}-\frac{1}{16}=\frac{1}{48} \quad \text{Ans.}$$

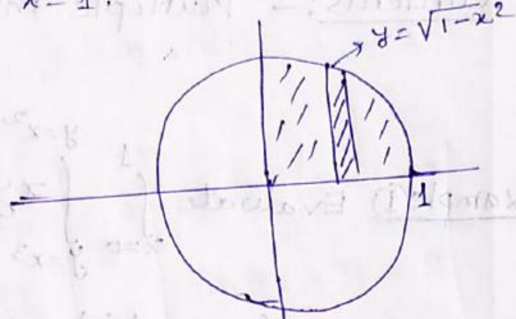
2.

Example (2) Write $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx$ as an integral over a region. Change the order of integration and evaluate.

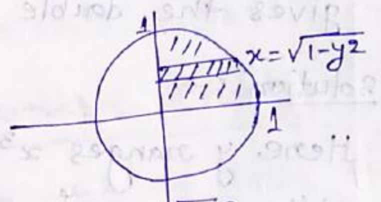
Solution: $\rightarrow$ $y=0$ to $y=\sqrt{1-x^2}$

$\Rightarrow x^2+y^2=1$

$x=0$, to $x=1$.



changing the order of integration



$$\therefore \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx = \int_{y=0}^{1} \int_{x=0}^{x=\sqrt{1-y^2}} \sqrt{1-y^2}\, dx\, dy$$

$$= \int_{y=0}^{1} \left[\sqrt{1-y^2} \cdot x \right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \int_{y=0}^{1} (1-y^2)\, dy = \left[y - \frac{y^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$ **Ans.**

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Example ③ Evaluate

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\, dy\, dx$$

Solution : →

$$\int_{x=0}^1 \int_{y=0}^{y=x} \int_{z=x^2+y^2}^2 dz\, dy\, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{y=x} \left[z\right]_{x^2+y^2}^2 dy\, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{y=x} (2-x^2-y^2)\, dy\, dx = \int_0^1 \left(2x - x^3 - \frac{x^3}{3}\right) dx$$

$$= 1 - \frac{1}{4} - \frac{1}{12} = \frac{2}{3} \text{ Ans.}$$

Remark: → The Jacobian determinant

If $x = x(u,v)$ & $y = y(u,v)$, then the Jacobian determinant is the determinant of the derivative matrix:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example: → Compute the Jacobian determinant of
$x = u + v, \; y = u - v$

Solution: →

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \quad \text{Ans.}$$

Example: → Compute the Jacobian determinant of $x = r\cos\theta, \; y = r\sin\theta$

Solution: →

$$\cancel{\frac{\partial(x,y)}{\partial(u,v)}} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

Remark: → (Change of variable formula)

Let D be the region in the xy plane that corresponds to a region D^* under a transformation in which x & y are functions of u and v. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

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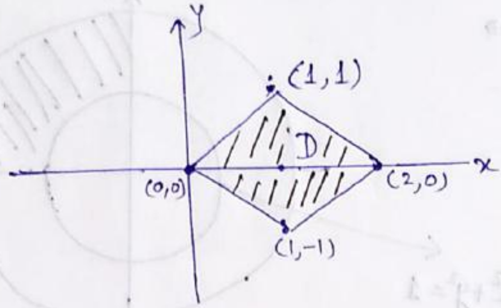
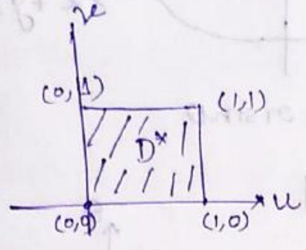
Example④ Evaluate $\iint_D (x^2-y^2)\,dx\,dy$ where D is the square with vertices $(0,0)$, $(1,-1)$, $(1,1)$ and $(2,0)$ using the change of variable $x = u+v$, $y = u-v$.

Solution :→





$$\left\{\begin{aligned} \because\ x &= u+v \\ \&\ y &= u-v \\ u &= \tfrac{1}{2}(x+y) \\ v &= \tfrac{1}{2}(x-y) \end{aligned}\right.$$

This transformation is linear & takes $(0,0)$ to $(0,0)$, $(1,0)$ to $(1,1)$, $(1,1)$ to $(2,0)$ and $(0,1)$ to $(1,-1)$, So it takes the parallelogram D^* to D.

The Integrand is

$$x^2-y^2 = (u+v)^2-(u-v)^2 = 4uv$$

$$\& \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

So, the change of variables formula gives

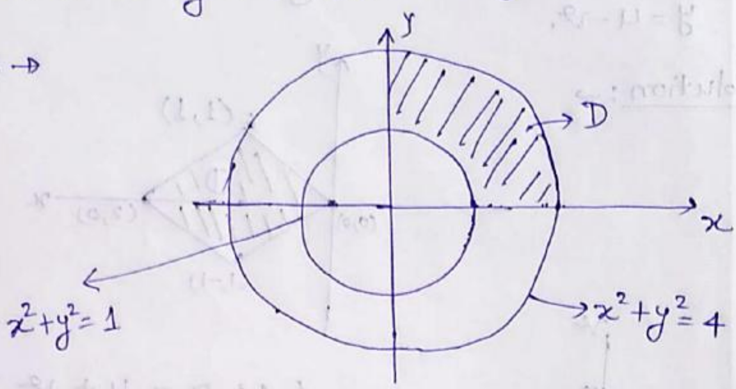
$$\iint_D (x^2-y^2)\,dx\,dy = \iint_{D^*} 4uv \cdot |-2|\,du\,dv$$

$$= 8\int_0^1\int_0^1 uv\,du\,dv = 8\int_0^1 u\,du \int_0^1 v\,dv$$

$$= 8\cdot\left[\frac{u^2}{2}\right]_0^1\cdot\left[\frac{v^2}{2}\right]_0^1 = 2 \quad \underline{\text{Ans.}}$$

Example (5) Evaluate $\iint_D \log(x^2+y^2)\,dx\,dy$, where D is the region in the first quadrant lying between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

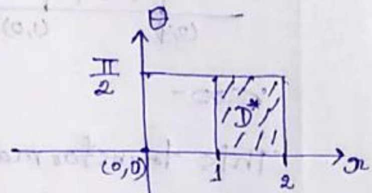
Solution: →



putting $x = r\cos\theta$ & $y = r\sin\theta$

then $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |r|$.

∴ The region D^* in the (r,θ) plane that corresponds to D is the rectangle $1 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$. Hence



$$\iint_D \log(x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=1}^{2} \log r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^2 2(\log r)\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}\cdot 2\cdot\log r - \frac{r^2}{2}\right)_{r=1}^{2} d\theta$$

$$= \int_0^{\pi/2}\left(4\log 2 - \frac{3}{2}\right) d\theta = \frac{\pi}{2}\left(4\log 2 - \frac{3}{2}\right)$$

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